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A NOTE ON THE ASYMPTOTIC STRESS FIELD OF A NON-UNIFORMLY PROPAGATING DYNAMIC CRACK

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Recent experimental evidence obtained by means of optical techniques, (e.g. the method of caustics [1,2] and the Coherent Gradient Sensor technique (CGS) [3]) have suggested that a truely transient high order expansion at the vicinity of dynamically moving crack tips is essential in the interpretation of experimental data in dynamic fracture. A higher order expansion for the first stress invarient, under plane stress conditions, is presented by Freund [4] (constant velocity transient growth) and by Freund and Rosakis [5] (non-uniform velocity growth). In this note, we present the basic steps for the derivation of the complete stress field at the vicinity of a mode I crack tip propagating with non-uniform velocity.

Consider a planar, mode I crack that grows through a two-dimensional, homogeneous, isotropic, linearly elastic solid, with a non-uniform speed $\upsilon(t)$, along the positive x_1 direction. (x_1,x_2) is a coordinate system which translates with the moving crack tip. In terms of the displacement potential functions $\Phi(x_1,x_2,t)$ and $\Psi(x_1,x_2,t)$, the equations of motion in the absence of body forces, can be expressed as [4]

$$\left(1-\frac{\upsilon^2(t)}{c_l^2}\right)\frac{\partial^2\Phi}{\partial x_1^2}+\frac{\partial^2\Phi}{\partial x_2^2}+\frac{\dot\upsilon(t)}{c_l^2}\frac{\partial\Phi}{\partial x_1}$$

$$+\frac{2\upsilon(t)}{c_l^2}\frac{\partial^2\Phi}{\partial x_1\partial t}-\frac{1}{c_l^2}\frac{\partial^2\Phi}{\partial t^2}=0 \tag{1}$$

$$\left(1-\frac{\upsilon^2(t)}{c_s^2}\right)\frac{\partial^2\Psi}{\partial x_1^2}+\frac{\partial^2\Psi}{\partial x_2^2}+\frac{\dot\upsilon(t)}{c_s^2}\frac{\partial\Psi}{\partial x_1}$$

$$+\frac{2\upsilon(t)}{c_s^2}\frac{\partial^2\Psi}{\partial x_1 \partial t}-\frac{1}{c_s^2}\frac{\partial^2\Psi}{\partial t^2}=0 \tag{2}$$

where c_l and c_s are the longitudinal and shear wave speeds of the elastic material, respectively. By assuming that Φ and Ψ can asymptotically be expressed as

$$\Phi(x_1,x_2,t)=\sum_{m=0}^{\infty}\in^{\frac{m+3}{2}}\Phi_m(\eta_1,\eta_2,t),$$

and

$$\Psi(x_1,x_2,t)=\sum_{m=0}^{\infty}\in^{\frac{m+3}{2}}\Psi_m(\eta_1,\eta_2,t),$$

as

$$r=\sqrt{x_1^2+x_2^2}\to 0,$$

where $\eta_\alpha = x_\alpha/\in$, $\alpha = 1,2$, and $\in$ is an arbitrary positive number. Then, (1) and (2) reduce to a series of coupled differential equations for $\Phi_m(\eta_1,\eta_2,t)$ and $\Psi_m(\eta_1,\eta_2,t)$ as follows:

$$\alpha_l^2(t)\frac{\partial^2\Phi_m}{\partial\eta_1^2}+\frac{\partial^2\Phi_m}{\partial\eta_2^2}=$$
$$-\frac{2\upsilon^{\frac{1}{2}}(t)}{c_l^2}\frac{\partial}{\partial t}\left\{\upsilon^{\frac{1}{2}}(t)\frac{\partial\Phi_{m-2}}{\partial\eta_1}\right\}+\frac{1}{c_l^2}\frac{\partial^2\Phi_{m-4}}{\partial t^2}, \tag{3}$$

and

$$\alpha_s^2(t)\frac{\partial^2\Psi_m}{\partial\eta_1^2}+\frac{\partial^2\Psi_m}{\partial\eta_2^2}=$$
$$-\frac{2\upsilon^{\frac{1}{2}}(t)}{c_s^2}\frac{\partial}{\partial t}\left\{\upsilon^{\frac{1}{2}}(t)\frac{\partial\Psi_{m-2}}{\partial\eta_1}\right\}+\frac{1}{c_s^2}\frac{\partial^2\Psi_{m-4}}{\partial t^2}, \tag{4}$$

where

$$\alpha_{l,s}^2(t)=1-\frac{\upsilon^2(t)}{c_{l,s}^2},$$

$$\Phi_k=\begin{cases}\Phi_k & \text{for} \quad k\geq 0\\ 0 & \text{for} \quad k<0\end{cases}$$

$$\Psi_k = \begin{cases} \Psi_k & \text{for} \quad k \geq 0 \\ 0 & \text{for} \quad k < 0 \end{cases}$$

If we further introduce the scaled polar coordinate systems (r_l, θ_l) and (r_s, θ_s) where

$$r_{l,s}^2(t) = \eta_1^2 + \alpha_{l,s}^2(t)\eta_2^2,$$

$$\theta_{l,s}(t) = \tan^{-1}\left\{\frac{\alpha_{l,s}(t)\eta_2}{\eta_1}\right\}$$

and also define $\hat{\Phi}_m$ and $\hat{\Psi}_m$ as

$$\Phi_m(\eta_1,\eta_2,t) = \hat{\Phi}_m(r_l,\theta_l,t),$$

$$\Psi_m(\eta_1,\eta_2,t) = \hat{\Psi}_m(r_s,\theta_s,t),$$

then the general solutions of the first three terms of Φ_m and Ψ_m in (3) and (4) are

$$\hat{\Phi}_0(r_l,\theta_l,t) = A_0(t) r_l^{\frac{3}{2}} \cos\frac{3\theta_l}{2}, \tag{5}$$

$$\hat{\Phi}_1(r_l,\theta_l,t) = A_1(t) r_l^2 \cos 2\theta_l, \tag{6}$$

$$\hat{\Phi}_2(r_l,\theta_l,t) = r_l^{\frac{5}{2}}\left\{\frac{1}{6}[D_l^1\{A_0(t)\} + \frac{1}{2}B_l(t)]\sin\frac{\theta_l}{2} - \frac{1}{8}B_l(t)\cos\frac{3\theta_l}{2} + A_2(t)\cos\frac{5\theta_l}{2}\right\}, \tag{7}$$

and

$$\hat{\Psi}_0(r_s,\theta_s,t) = B_0(t) r_s^{\frac{3}{2}} \sin\frac{3\theta_s}{2}, \tag{8}$$

$$\hat{\Psi}_1(r_s,\theta_s,t) = B_1(t) r_s^2 \sin 2\theta_s, \tag{9}$$

$$\hat{\Psi}_2(r_s,\theta_s,t) = r_s^{\frac{5}{2}}\left\{\frac{1}{6}[D_s^1\{B_0(t)\} + \frac{1}{2}B_s(t)]\sin\frac{\theta_s}{2} + \frac{1}{8}B_s(t)\sin\frac{3\theta_s}{2} + B_2(t)\sin\frac{5\theta_s}{2}\right\}, \tag{10}$$

where

$$A_0(t) = \frac{4}{3\mu\sqrt{2\pi}} \frac{1+\alpha_s^2}{D(v)} K_I^d(t) \ ,$$

$$\begin{aligned} D_l^1\{A_0(t)\} &= -\frac{3v^{\frac{1}{2}}(t)}{\alpha_l^2 c_l^2} \frac{d}{dt} \left\{ v^{\frac{1}{2}}(t) A_0(t) \right\} \\ &= -\frac{4v^{\frac{1}{2}}(t)}{\mu\sqrt{2\pi}\alpha_l^2 c_l^2} \frac{d}{dt} \left\{ v^{\frac{1}{2}}(t) \frac{1+\alpha_s^2}{D(v)} K_I^d(t) \right\} \ , \end{aligned}$$

$$\begin{aligned} B_l(t) &= \frac{3v^2(t)}{2\alpha_l^4 c_l^4} A_0(t) \frac{dv(t)}{dt} \\ &= \frac{2v^2(t)}{\mu\sqrt{2\pi}\alpha_l^4 c_l^4} \frac{1+\alpha_s^2}{D(v)} K_I^d(t) \frac{dv(t)}{dt} \ , \end{aligned}$$

and

$$B_0(t) = -\frac{4}{3\mu\sqrt{2\pi}} \frac{2\alpha_l}{D(v)} K_I^d(t) \ ,$$

$$\begin{aligned} D_s^1\{B_0(t)\} &= -\frac{3v^{\frac{1}{2}}(t)}{\alpha_s^2 c_s^2} \frac{d}{dt} \left\{ v^{\frac{1}{2}}(t) B_0(t) \right\} \\ &= \frac{4v^{\frac{1}{2}}(t)}{\mu\sqrt{2\pi}\alpha_s^2 c_s^2} \frac{d}{dt} \left\{ v^{\frac{1}{2}}(t) \frac{2\alpha_l}{D(v)} K_I^d(t) \right\} \ , \end{aligned}$$

$$\begin{aligned} B_s(t) &= \frac{3v^2(t)}{2\alpha_s^4 c_s^4} B_0(t) \frac{dv(t)}{dt} \\ &= -\frac{2v^2(t)}{\mu\sqrt{2\pi}\alpha_s^4 c_s^4} \frac{2\alpha_l}{D(v)} K_I^d(t) \frac{dv(t)}{dt} \ , \end{aligned}$$

and

$$D(v) = 4\alpha_l\alpha_s - (1+\alpha_s^2)^2 \ .$$

Recall that

$$\begin{aligned} \frac{\sigma_{11}}{\mu} &= \left(\frac{1-\alpha_s^2}{1-\alpha_l^2}\right) \frac{\partial^2\Phi}{\partial x_1^2} + \left(\frac{2\alpha_l^2-\alpha_s^2-1}{1-\alpha_l^2}\right) \frac{\partial^2\Phi}{\partial x_2^2} + 2\frac{\partial^2\Psi}{\partial x_1 \partial x_2} \ , \\ \frac{\sigma_{22}}{\mu} &= \left(\frac{2\alpha_l^2-\alpha_s^2-1}{1-\alpha_l^2}\right) \frac{\partial^2\Phi}{\partial x_1^2} + \left(\frac{1-\alpha_s^2}{1-\alpha_l^2}\right) \frac{\partial^2\Phi}{\partial x_2^2} - 2\frac{\partial^2\Psi}{\partial x_1 \partial x_2} \ , \end{aligned}$$

$$\frac{\sigma_{12}}{\mu} = 2\frac{\partial^2\Phi}{\partial x_1\partial x_2} + \frac{\partial^2\Psi}{\partial x_2^2} - \frac{\partial^2\Psi}{\partial x_1^2} ,$$

then, by setting $\in = 1$, we can get the asymptotic stress field around the tip of a non-uniformly propagating crack

$$\begin{aligned}
\frac{\sigma_{11}}{\mu} =& \frac{1}{\mu\sqrt{2\pi}}\frac{K_I^d(t)}{D(v)}\left\{(1+2\alpha_l^2-\alpha_s^2)(1+\alpha_s^2)r_l^{-\frac{1}{2}}\cos\frac{\theta_l}{2} - 4\alpha_l\alpha_s r_s^{-\frac{1}{2}}\cos\frac{\theta_s}{2}\right\} \\
&+2(1+2\alpha_l^2-\alpha_s^2)A_1(t) + 4\alpha_s B_1(t) \\
&+\left\{D_l^1\{A_0(t)\}\left[\frac{(1+\alpha_l^2)(\alpha_l^2-\alpha_s^2)+(1-\alpha_l^2)^2}{2(1-\alpha_l^2)}\cos\frac{\theta_l}{2} + \frac{1+2\alpha_l^2-\alpha_s^2}{8}\cos\frac{3\theta_l}{2}\right]\right. \\
&+\frac{1}{2}B_l(t)\left[\frac{(2\alpha_l^2-\alpha_s^2-1)\alpha_l^2}{1-\alpha_l^2}\cos\frac{\theta_l}{2} + \{\frac{5(2\alpha_l^2-\alpha_s^2)}{8} - \frac{\alpha_l^2-\alpha_s^2}{1-\alpha_l^2} - \frac{3}{8}\}\cos\frac{3\theta_l}{2}\right. \\
&\left.\left.+\frac{1+2\alpha_l^2-\alpha_s^2}{16}\cos\frac{7\theta_l}{2}\right] + \frac{15(1+2\alpha_l^2-\alpha_s^2)}{4}A_2(t)\cos\frac{\theta_l}{2}\right\}r_l^{\frac{1}{2}} \\
&+2\alpha_s\left\{\frac{1}{8}D_s^1\{B_0(t)\}\cos\frac{3\theta_s}{2} + \frac{1}{4}B_s(t)\left(\cos\frac{\theta_s}{2} + \frac{1}{4}\cos\frac{3\theta_s}{2} + \frac{1}{8}\cos\frac{7\theta_s}{2}\right)\right. \\
&\left.+\frac{15}{4}B_2(t)\cos\frac{\theta_s}{2}\right\}r_s^{\frac{1}{2}} + O(r_{l,s}) ,
\end{aligned} \tag{11}$$

$$\begin{aligned}
\frac{\sigma_{22}}{\mu} =& \frac{1}{\mu\sqrt{2\pi}}\frac{K_I^d(t)}{D(v)}\left\{-(1+\alpha_s^2)^2 r_l^{-\frac{1}{2}}\cos\frac{\theta_l}{2} + 4\alpha_l\alpha_s r_s^{-\frac{1}{2}}\cos\frac{\theta_s}{2}\right\} \\
&-2(1+\alpha_s^2)A_1(t) - 4\alpha_s B_1(t) \\
&\left\{D_l^1\{A_0(t)\}\left[\{\frac{(1+\alpha_l^2)(\alpha_l^2-\alpha_s^2)-(1-\alpha_l^2)^2}{2(1-\alpha_l^2)}\}\cos\frac{\theta_l}{2} - \frac{1+\alpha_s^2}{8}\cos\frac{3\theta_l}{2}\right]\right. \\
&+\frac{1}{2}B_l(t)\left[\frac{(1-\alpha_s^2)\alpha_l^2}{1-\alpha_l^2}\cos\frac{\theta_l}{2} + \{\frac{3-5\alpha_s^2}{8} - \frac{\alpha_l^2-\alpha_s^2}{1-\alpha_l^2}\}\cos\frac{3\theta_l}{2} - \frac{1+\alpha_s^2}{16}\cos\frac{7\theta_l}{2}\right] \\
&\left.-\frac{15(1+\alpha_s^2)}{4}A_2(t)\cos\frac{\theta_l}{2}\right\}r_l^{\frac{1}{2}}
\end{aligned}$$

$$-2\alpha_s\left\{\frac{1}{8}D_s^1\{B_0(t)\}\cos\frac{3\theta_s}{2}+\frac{1}{4}B_s(t)\left(\cos\frac{\theta_s}{2}+\frac{1}{4}\cos\frac{3\theta_s}{2}+\frac{1}{8}\cos\frac{7\theta_s}{2}\right)\right.$$
$$\left.+\frac{15}{4}B_2(t)\cos\frac{\theta_s}{2}\right\}r_s^{\frac{1}{2}}+O(r_{l,s})\ , \tag{12}$$

$$\begin{aligned}\frac{\sigma_{12}}{\mu} &= \frac{1}{\mu\sqrt{2\pi}}\frac{2\alpha_l(1+\alpha_s^2)}{D(v)}K_I^d(t)\left(r_l^{-\frac{1}{2}}\sin\frac{\theta_l}{2}-r_s^{-\frac{1}{2}}\sin\frac{\theta_s}{2}\right)\\ &+2\alpha_l\left\{\frac{1}{8}D_l^1\{A_0(t)\}\sin\frac{3\theta_l}{2}-\frac{1}{4}B_l(t)\left(\sin\frac{\theta_l}{2}-\frac{1}{4}\sin\frac{3\theta_l}{2}-\frac{1}{8}\sin\frac{7\theta_l}{2}\right)\right.\\ &\left.-\frac{15}{4}A_2(t)\sin\frac{\theta_l}{2}\right\}r_l^{\frac{1}{2}}\\ &-\left\{\frac{1}{2}D_s^1\{B_0(t)\}\left[(1-\alpha_s^2)\sin\frac{\theta_s}{2}-\frac{1+\alpha_s^2}{4}\sin\frac{3\theta_s}{2}\right]\right.\\ &-\frac{1}{2}B_s(t)\left[\alpha_s^2\sin\frac{\theta_s}{2}+\frac{5\alpha_s^2-3}{8}\sin\frac{3\theta_s}{2}+\frac{1+\alpha_s^2}{16}\sin\frac{7\theta_s}{2}\right]\\ &\left.+\frac{15(1+\alpha_s^2)}{4})B_2(t)\sin\frac{\theta_s}{2}\right\}r_s^{\frac{1}{2}}+O(r_{l,s})\ .\end{aligned} \tag{13}$$

In the expressions above, $A_0(t)$ and $B_0(t)$ are determined by the dynamic stress intensity factor history, $K^d_I(t)$, and the propagating speed of the crack tip, $\upsilon(t)$. $D^1_l\{A_0(t)\}$ and $D^1_s\{B_0(t)\}$ depend not only on $K^d_I(t)$ and $\upsilon(t)$, but also on the time derivatives of these quantities. Besides $K^d_I(t)$ and $\upsilon(t)$, $B_l(t)$ and $B_s(t)$ also depend on the acceleration of the crack. The coefficients $A_1(t)$,$A_2(t)$,$B_1(t)$, and $B_2(t)$ are undetermined by the asymptotic analysis. Their values can be determined for particular initial/boundary value problems. Since the crack surfaces are traction free, as $\theta_l = \theta_s = \pm\pi$, we should have $\sigma_{22} = \sigma_{12} = 0$. From (12) and (13), we get

$$(1+\alpha_s^2)A_1(t)+2\alpha_s B_1(t)=0, \tag{14}$$

and

$$30\{2\alpha_l A_2(t)+(1+\alpha_s^2)B_2(t)\}=-\{2\alpha_l D_l^1\{A_0(t)\}+(5-3\alpha_s^2)D_s^1\{B_0(t)\}\}$$

$$-\frac{1}{4}\{22\alpha_l B_l(t)-5(1+\alpha_s^2)B_s(t)\}, \tag{15}$$

which means that $A_1(t)$ and $B_1(t)$, $A_2(t)$ and $B_2(t)$ are not independent parameters, they are related by (14) and (15), respectively. As a matter of fact, (15) shows that the combination of $A_2(t)$ and $B_2(t)$ is related to the stress intensity factor history.

If the crack tip speed $\upsilon(t)$ is a constant, i.e. $\dot{\upsilon}(t) = 0$, and therefore, $B_l(t) = B_s(t) = 0$, we can obtain the asymptotic stress field corresponding to transient crack growth under constant velocity and varying stress intensity factor (see Freund and Rosakis [5]). A classical example of such a transient crack problem is the one analyzed by Broberg (see Freund [4], Freund and Rosakis [5]). Furthermore, if the time derivative of the dynamic stress intensity factor, $\dot{K}^d_I(t)$, is also zero, $D^1_l\{A_0(t)\}$ and $D^1_s\{B_0(t)\}$ will be zero, then we obtain the familiar results of the asymptotic stress field for steady state up to three terms.

Finally, we are in the process of applying the above stress representation to the analysis of various experimental methods, such as the caustic method, photoelasticity, and the more recently developed coherent gradient sensing method (CGS), and to investigate the influence of the dynamic transient effects on the interpretation of the experimental patterns.

REFERENCES

[1] S. Krishnaswamy and A.J. Rosakis, "On the extent of dominance of asymptotic elastodynamic crack tip fields, Part I: An experimental study using bifocal caustics", Caltech Report SM 88-21.

[2] A.J. Rosakis, S. Krishnaswamy and H.V. Tippur, *Optics and Lasers in Engineering* 13 (1990) 183-210.

[3] S. Krishnaswamy, H.V. Tippur and A.J. Rosakis, "Measurement of transient crack tip deformation fields using the method of coherent gradient sensing", Caltech Report SM-90-1 (1990).

[4] L.B. Freund, *Dynamic Fracture Mechanics*, Cambridge University Press (1990).

[5] L.B. Freund and A.J. Rosakis, in *11th National Congress of Applied Mechanics,* Tucson, Arizona, May 1990.

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